Libyan Journal of Medical and Applied Sciences
LJMAS
Online ISSN: 3006-1113
Volume 3, Issue 4, 2025, Page No: 48-54
Website: https://ljmas.com/index.php/journal/index
DOI: https://doi.org/10.64943/ljmas.v3i4.190

Analytical Investigation of Positive Continuous Solutions for Nonlinear Quadratic Integral Equations of Hammerstein Type via Fixed-Point Methodological Approaches

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Received: August 05, 2025 Accepted: October 05, 2025 Published: October 10, 2025

Cite this article as: M, K, Mohamed., E, M, Ali. (2025) Analytical Investigation of Positive Continuous Solutions for Nonlinear Quadratic Integral Equations of Hammerstein Type via Fixed-Point Methodological Approaches. Libyan Journal of Medical and Applied Sciences (LJMAS). 2025;3(4):48-54.

Abstract:
This study establishes rigorous existence criteria for positive continuous solutions to a significant class of nonlinear quadratic integral equations of Hammerstein type. The equation under consideration takes the form:

$$y(\tau) = b(\tau) + \int_0^\tau H_1(\tau,\sigma) f_1(\sigma, y(\sigma)) d\sigma \int_0^\tau H_2(\tau,\sigma) f_2(\sigma, y(\sigma)) d\sigma, \tau \in [0,\Theta]$$

Methodologically, we employ Schauder's fixed-point theorem as our principal analytical tool to derive the central existence result. Furthermore, under appropriately formulated monotonicity constraints, we establish the existence of both maximal and minimal solutions. These theoretical contributions extend the existing mathematical literature on quadratic integral equations by introducing novel methodological frameworks and expanding the applicable domain of existence criteria for this important category of nonlinear functional equations.

Keywords: Nonlinear Quadratic Integral Equations, Hammerstein-Type Operators, Carathéodory-Class Functions, Monotonic Operators, Extremal Solutions, Lebesgue Integration Theory, Schauder Fixed-Point Methodology.

الدراسة التحليلية للحلول المستمرة الموجبة لمعادلات تكاملية تربيعية غير خطية من نوع هامرستين باستخدام مناهج النقطة الثابتة

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الملخص
تُحدِّد هذه الدراسة معايير وجودٍ رياضية صارمة للحلول المستمرة الموجبة لفئةٍ ذات أهمية من المعادلات التكاملية التربيعية غير الخطية من نوع هامرستين. وتأخذ المعادلة موضوع البحث الصيغة التالية:

$$y(\tau) = b(\tau) + \int_0^\tau H_1(\tau,\sigma) f_1(\sigma, y(\sigma)) d\sigma \int_0^\tau H_2(\tau,\sigma) f_2(\sigma, y(\sigma)) d\sigma, \tau \in [0,\Theta]$$

من الناحية المنهجية، نعتمد مبرهنة شاودر للنقطة الثابتة كأداة تحليلية رئيسية لاستنتاج نتيجة الوجود الأساسية. وعلاوةً على ذلك، وبفرض قيود رتوبة (أو تتابعية) مناسبة، نبرهن على وجود كلٍّ من الحل الأقصى والحل الأدنى. وتُعدُّ هذه المساهمات النظرية امتدادًا للبحوث الرياضية السابقة حول المعادلات التكاملية التربيعية، من خلال تقديم أطر منهجية جديدة وتوسيع نطاق تطبيق معايير الوجود ليشمل فئةً مهمة من المعادلات الدالية غير الخطية.

الكلمات المفتاحية: المعادلات التكاملية التربيعية غير الخطية، مؤثّرات من نوع هامرستين، دوال من صنف كاراتيودوري، المؤثّرات الرتيبة، الحلول القصوى، نظرية التكامل ليبغ، منهجية شاودر للنقطة الثابتة.

Introduction

The analytical study of integral equations occupies a central position in modern mathematical analysis, with implications that extend deeply into engineering applications, physical modeling, and various branches of applied mathematics. Historically, the emergence of integral equations as a distinct mathematical discipline can be traced to foundational work by Du Bois-Reymond in the late nineteenth century [1], which subsequently stimulated extensive research interest within the mathematical community. Contemporary scholarship continues to develop systematic classification frameworks for specialized categories of these equations, as evidenced by comprehensive surveys in the field [2], reflecting their enduring theoretical importance.

Within this broad mathematical landscape, quadratic integral equations represent a particularly intriguing subclass characterized by their inherent nonlinear structural properties. Early mathematical investigations in this area emerged from Chandrasekhar's pioneering work on radiative transfer phenomena [3], with subsequent applications materializing across diverse domains including kinetic gas theory [4], neutron transport modeling [5], and vehicular traffic flow analysis [6]. The persistent appearance of these equations across such varied applied contexts underscores their fundamental mathematical significance.

Our present investigation focuses specifically on establishing comprehensive existence criteria for positive continuous solutions associated with Hammerstein-type quadratic integral equations, while simultaneously developing methodological frameworks for characterizing extremal solutions under appropriate monotonicity conditions. This research endeavor contributes to the existing mathematical literature through the introduction of novel analytical techniques and the expansion of applicable existence domains for this important class of nonlinear functional equations. By addressing both general existence and the more specialized theory of extremal solutions, we aim to provide a more complete theoretical understanding of solution behavior for these mathematically rich and practically relevant equations.

Theoretical Foundations and Mathematical Preliminaries

This section establishes the fundamental mathematical framework and auxiliary results that underpin our subsequent analytical developments. We consider the compact temporal interval $J = [0, \Theta]$, and denote by $L^1 = L^1[0, \Theta]$ the Banach space comprising Lebesgue integrable functions defined on J.

To facilitate our investigation, we introduce the following foundational assumptions:

(A1) The function $b: J = [0, \Theta] \to \mathbb{R}^+$ exhibits continuity throughout its domain, with the boundedness condition $b = \sup_{\tau \in [0,\Theta]} |b(\tau)| < \infty$ being satisfied. (A2) For each index $i = 1,2$, the following conditions hold:

- The kernel functions $H_i: [0, \Theta] \times [0, \Theta] \to \mathbb{R}^+$ demonstrate continuity over their entire domain.
- The nonlinear component functions $f_i: [0, \Theta] \times \mathbb{R}^+ \to \mathbb{R}^+$ belong to the Carathéodory class, being measurable in the variable σ for every fixed $y \in \mathbb{R}^+$ and continuous in y for almost every $\sigma \in [0, \Theta]$.
- There exist Lebesgue integrable functions $n_i(\sigma) \in L^1[0, \Theta]$ satisfying the bounding condition:

$$|f_i(\sigma, y)| \le n_i(\sigma), i = 1,2$$

with the additional constraint that $\int_0^\tau n_i(\sigma) d\sigma \le N_i, i = 1,2$, for all $\tau \in [0, \Theta]$.

- The functions $f_i, i = 1,2$, exhibit monotonic nonincreasing behavior with respect to the temporal variable $\tau \in [0, \Theta]$. (A3) For the analysis of extremal solutions, we additionally assume that the functions $f_i(\sigma, y), i = 1,2$ demonstrate monotonic non-decreasing behavior in the variable y for each fixed $\sigma \in [0, \Theta]$.

Remark 2.1. The distinction between assumption (A2) and (A3) is crucial. Condition (A2) concerns monotonicity with respect to the temporal variable τ and is used primarily to establish the existence of positive continuous solutions via Schauder's fixed point theorem. Condition (A3), by contrast, addresses monotonicity in the solution variable y and is essential for the development of extremal solution theory in Section 4. This separation clarifies the distinct roles these monotonicity conditions play in our analysis. The subsequent fundamental theorems play instrumental roles in establishing our principal results, providing the theoretical scaffolding upon which our analysis rests.

Consider a convex subset Ψ of a Banach space $\mathcal{B}$. If the operator $\mathcal{F}: \Psi \to \Psi$ demonstrates both compactness and continuity properties, then $\mathcal{F}$ admits at least one fixed point within the domain Ψ [7]. Let $\mho$ represent a compact metric space and $C(\mho)$ denote the Banach space consisting of real or complex-valued continuous functions equipped with the supremum norm $\|f\| = \max_{\tau \in \mho} |f(\tau)|$. If the sequence $\aleph = \{f_k\}$ in $C(\mho)$ exhibits both uniform boundedness and equicontinuity, then the closure of $\aleph$ constitutes a compact set [8].

Consider a sequence of functions $\{\ell_k\}$ converging to a limit function ℓ on the domain A, with the bounding condition $|\ell_k(\tau)| \le \emptyset(\tau)$ holding for $\tau \in A, k = 1,2,3, \ldots$, where $\emptyset$ represents an integrable function over A. Under these conditions, the limit function ℓ maintains integrability over A and satisfies:

$$\lim_{k\to\infty} \int_A \ell_k(\tau)d\mu = \int_A \ell(\tau)d\mu$$

The following formal definition, originally introduced by Lakshmikantham and colleagues [9], proves essential for our subsequent analytical developments concerning extremal solutions. [Maximal and Minimal Solutions [9]] Let $c(\tau)$ represent a solution of the quadratic integral equation under investigation. This solution $c(\tau)$ qualifies as a maximal solution if every alternative solution $y(\tau)$ of the equation satisfies the inequality $y(\tau) \le c(\tau)$ for all $\tau \in [0,\Theta]$. Conversely, a minimal solution $d(\tau)$ may be defined through reversal of the inequality, specifically requiring $y(\tau) \ge d(\tau)$ for $\tau \in [0,\Theta]$.

Existence of Positive Continuous Solutions

This section presents the central theoretical contributions of our investigation, establishing existence criteria for positive continuous solutions. Let $C = C[0,\Theta]$ represent the Banach space comprising continuous functions defined on the interval J, and define the solution candidate set Ω through the specification:

$$\Omega = \{y \in C: 0 \le y \le R\} \subset C[0,\Theta], \text{ where } R = b + N_1N_2\|H_1\|\|H_2\|$$

with the kernel bounds given by $\|H_i\| = \sup_{(\tau,\sigma)\in[0,\Theta]\times[0,\Theta]} |H_i(\tau,\sigma)|$, for $i = 1,2$. The set Ω manifestly satisfies the properties of closure, convexity, boundedness, and non-emptiness, thereby constituting an appropriate domain for fixed-point analysis.

Under the validity of assumptions (A1) and (A2), the Hammerstein quadratic integral equation

$$y(\tau) = b(\tau) + \int_0^\tau H_1(\tau,\sigma)f_1(\sigma,y(\sigma))d\sigma \int_0^\tau H_2(\tau,\sigma)f_2(\sigma,y(\sigma))d\sigma, \tau \in [0,\Theta]$$

admits at least one positive continuous solution $y \in C[0,\Theta]$.

Proof. To establish this fundamental result, we introduce the operator mapping Φ defined by:

$$\Phi y(\tau) = b(\tau) + \int_0^\tau H_1(\tau,\sigma)f_1(\sigma,y(\sigma))d\sigma \int_0^\tau H_2(\tau,\sigma)f_2(\sigma,y(\sigma))d\sigma$$

Consider an arbitrary element $y \in \Omega$. The following bounding relationships emerge through systematic analysis:

$$\begin{aligned}|\Phi y(\tau)| &= \left| b(\tau) + \int_0^\tau H_1(\tau,\sigma)f_1(\sigma,y(\sigma))d\sigma \int_0^\tau H_2(\tau,\sigma)f_2(\sigma,y(\sigma))d\sigma \right| \\ &\le |b(\tau)| + \left|\int_0^\tau H_1(\tau,\sigma)f_1(\sigma,y(\sigma))d\sigma\right| \left|\int_0^\tau H_2(\tau,\sigma)f_2(\sigma,y(\sigma))d\sigma\right| \\ &\le |b(\tau)| + \int_0^\tau |H_1(\tau,\sigma)f_1(\sigma,y(\sigma))|d\sigma \int_0^\tau |H_2(\tau,\sigma)f_2(\sigma,y(\sigma))|d\sigma \\ &\le |b(\tau)| + \|H_1\| \int_0^\tau n_1(\sigma)d\sigma \cdot \|H_2\| \int_0^\tau n_2(\sigma)d\sigma \\ &\le b + N_1N_2\|H_1\|\|H_2\| = R.\end{aligned}$$

Furthermore, the positivity condition $b(\tau) > 0$ combined with the non-negativity of the integral components ensures that $\Phi y(\tau) \ge b(\tau) > 0$. These collective observations demonstrate that $\Phi y \in \Omega$ and that the image set $\{\Phi(y)\}$ maintains uniform boundedness.

To establish the equicontinuity property, consider temporal parameters $\tau_1, \tau_2 \in [0,\Theta]$ with $\tau_1 < \tau_2$ and $|\tau_2 - \tau_1| \le \delta$. Introducing the auxiliary functionals:

$$A(\tau) = \int_0^\tau H_1(\tau,\sigma)f_1(\sigma,y(\sigma))d\sigma$$

$$B(\tau) = \int_0^\tau H_2(\tau,\sigma)f_2(\sigma,y(\sigma))d\sigma$$

we observe the bounds $|A(\tau)| \le C_1 = \|H_1\|N_1$ and $|B(\tau)| \le C_2 = \|H_2\|N_2$. The difference analysis yields:

$$\begin{aligned}|\Phi y(\tau_2) - \Phi y(\tau_1)| &\le |b(\tau_2) - b(\tau_1)| + |A(\tau_2)B(\tau_2) - A(\tau_1)B(\tau_1)| \\ &\le |b(\tau_2) - b(\tau_1)| + C_2|A(\tau_2) - A(\tau_1)| + C_1|B(\tau_2) - B(\tau_1)|.\end{aligned}$$

Proceeding with the detailed estimation:

$$|A(\tau_2) - A(\tau_1)| \le \int_0^{\tau_1} |H_1(\tau_2,\sigma) - H_1(\tau_1,\sigma)||f_1(\sigma,y(\sigma))|d\sigma$$
$$+ \int_{\tau_1}^{\tau_2} |H_1(\tau_2,\sigma)||f_1(\sigma,y(\sigma))|d\sigma$$
$$\le \int_0^{\tau_1} |H_1(\tau_2,\sigma) - H_1(\tau_1,\sigma)|n_1(\sigma)d\sigma + \|H_1\| \int_{\tau_1}^{\tau_2} n_1(\sigma)d\sigma$$

The uniform continuity property of H_1 over the compact domain $[0,\Theta] \times [0,\Theta]$ guarantees that for any $\epsilon > 0$, there exists $\delta_1 > 0$ such that $|\tau_2 - \tau_1| < \delta_1$ implies $| H_1(\tau_2,\sigma) - H_1(\tau_1,\sigma) | < \epsilon$ uniformly in σ. Additionally, the absolute continuity of the integral $\int_0^\tau n_1(\sigma)d\sigma$ ensures the existence of $\delta_2 > 0$ such that $|\tau_2 - \tau_1| < \delta_2$ yields $\int_{\tau_1}^{\tau_2} n_1(\sigma)d\sigma < \epsilon$. Consequently, for $|\tau_2 - \tau_1| < \min(\delta_1,\delta_2)$:

$$|A(\tau_2) - A(\tau_1)| \le \epsilon N_1 + \|H_1\|\epsilon = \epsilon(N_1 + \|H_1\|)$$

Through analogous reasoning, $|B(\tau_2) - B(\tau_1)| \le \epsilon(N_2 + \|H_2\|)$. Combining these estimates produces:

$$|\Phi y(\tau_2) - \Phi y(\tau_1)| \le |b(\tau_2) - b(\tau_1)| + C_2\epsilon(N_1 + \|H_1\|) + C_1\epsilon(N_2 + \|H_2\|).$$

The uniform continuity of b enables selection of sufficiently small ϵ to render the righthand side arbitrarily small, thereby establishing the equicontinuity of $\{\Phi(y)\}$ throughout $[0,\Theta]$. Application of the Arzela-Ascoli theorem consequently verifies the compactness property of Φ.

To demonstrate continuity of the mapping $\Phi: \Omega \to \Omega$, consider a sequence $\{y_k\} \subset \Omega$ converging uniformly to y. The operator representation becomes:

$$\Phi y_k(\tau) = b(\tau) + \int_0^\tau H_1(\tau,\sigma)f_1(\sigma,y_k(\sigma))d\sigma \int_0^\tau H_2(\tau,\sigma)f_2(\sigma,y_k(\sigma))d\sigma$$

The point wise convergence $f_i(\sigma,y_k(\sigma)) \to f_i(\sigma,y(\sigma)), i = 1,2,$ coupled with the domination condition $|f_i(\sigma,y_k(\sigma))| \le n_i(\sigma), i = 1,2,$ permits application of the Lebesgue dominated convergence theorem, yielding:

$$\lim_{k\to\infty} \int_0^\tau H_1(\tau,\sigma)f_1(\sigma,y_k(\sigma))d\sigma = \int_0^\tau H_1(\tau,\sigma)f_1(\sigma,y(\sigma))d\sigma$$

with analogous convergence for the second integral. This establishes uniform convergence $\Phi y_k(\tau) \to \Phi y(\tau)$, thereby verifying the continuity of Φ.

Having satisfied all requisite conditions of Schauder's fixed-point theorem, the mapping Φ necessarily possesses at least one fixed point within $C[0,\Theta]$, which corresponds to a positive continuous solution of the integral equation under investigation.

Assuming the functions $f_1, f_2: [0,\Theta] \times \mathbb{R}^+ \to \mathbb{R}^+$exhibit L^1-Carathéodory characteristics with the bounding conditions $|f_i| \le n_i, i = 1,2,$ and the kernel functions H_1, H_2 maintain continuity throughout $[0,\Theta] \times [0,\Theta]$, then the integral equation admits at least one positive continuous solution.

Extremal Solutions Under Monotonicity Conditions

This section extends our analysis to establish the existence of maximal and minimal solutions under appropriate monotonicity constraints. We begin with a comparative lemma that plays a crucial role in our extremal solution theory.

Let the functions $f_i(\sigma,y), i = 1,2$ satisfy assumptions (A2) and (A3), and consider two continuous functions $y(\tau), z(\tau)$ defined on $[0,\Theta]$ fulfilling the inequalities:

$$y(\tau) \le b(\tau) + \int_0^\tau H_1(\tau,\sigma)f_1(\sigma,y(\sigma))d\sigma \int_0^\tau H_2(\tau,\sigma)f_2(\sigma,y(\sigma))d\sigma, \quad \tau \in [0,\Theta]$$
$$z(\tau) \ge b(\tau) + \int_0^\tau H_1(\tau,\sigma)f_1(\sigma,z(\sigma))d\sigma \int_0^\tau H_2(\tau,\sigma)f_2(\sigma,z(\sigma))d\sigma, \quad \tau \in [0,\Theta]$$

with at least one inequality maintaining strictness. Then:

$$y(\tau) < z(\tau), \tau > 0.$$

Proof. Proceeding by contradiction, suppose the conclusion fails. Then there exists a parameter value $\tau_1 > 0$ such that $y(\tau_1) = z(\tau_1)$ while $y(\tau) < z(\tau)$ for $0 < \tau < \tau_1$. The monotonicity properties of f_1, f_2 in the variable y (Assumption A3) generate the relationships:

$$y(\tau_1) \le b(\tau_1) + \int_0^{\tau_1} H_1(\tau_1,\sigma)f_1(\sigma,y(\sigma))d\sigma \int_0^{\tau_1} H_2(\tau_1,\sigma)f_2(\sigma,y(\sigma))d\sigma$$
$$< b(\tau_1) + \int_0^{\tau_1} H_1(\tau_1,\sigma)f_1(\sigma,z(\sigma))d\sigma \int_0^{\tau_1} H_2(\tau_1,\sigma)f_2(\sigma,z(\sigma))d\sigma \le z(\tau_1)$$

which contradicts the assumption $y(\tau_1) = z(\tau_1)$. Consequently, the strict inequality $y(\tau) < z(\tau)$ must hold for all $\tau > 0$. □

Under the validity of **assumptions** (A1), (A2), and (A3), the integral equation admits both maximal and minimal solutions.

Proof. The existence of the maximal solution is established first. For an arbitrary parameter $\epsilon > 0$, consider the modified equation:

$$y_\epsilon(\tau) = b(\tau) + \int_0^\tau H_1(\tau,\sigma) f_{1\epsilon}(\sigma, y_\epsilon(\sigma)) d\sigma \int_0^\tau H_2(\tau,\sigma) f_{2\epsilon}(\sigma, y_\epsilon(\sigma)) d\sigma, \tau \in [0,\Theta]$$

where the modified nonlinearities are defined by $f_{i\epsilon}(\sigma, y_\epsilon(\sigma)) = f_i(\sigma, y_\epsilon(\sigma)) + \epsilon, i = 1,2$. These modified functions $f_{i\epsilon}(\sigma, y_\epsilon(\sigma)), i = 1,2$ maintain L^1-Carathéodory properties, ensuring existence of solutions in $C[0,\Theta]$. For parameters ϵ_1, ϵ_2 satisfying $0 < \epsilon_2 < \epsilon_1 < \epsilon$, the representations become:

$$\begin{aligned} y_{\epsilon_2}(\tau) &= b(\tau) + \int_0^\tau H_1(\tau,\sigma) f_{1\epsilon_2}\big(\sigma, y_{\epsilon_2}(\sigma)\big) d\sigma \int_0^\tau H_2(\tau,\sigma) f_{2\epsilon_2}\big(\sigma, y_{\epsilon_2}(\sigma)\big) d\sigma \\ &= b(\tau) + \int_0^\tau H_1(\tau,\sigma)\big(f_1\big(\sigma, y_{\epsilon_2}(\sigma)\big) + \epsilon_2\big) d\sigma \int_0^\tau H_2(\tau,\sigma)\big(f_2\big(\sigma, y_{\epsilon_2}(\sigma)\big) + \epsilon_2\big) d\sigma \end{aligned}$$

with analogous expressions for $y_{\epsilon_1}(\tau)$. Application of Lemma 4 yields $y_{\epsilon_2}(\tau) < y_{\epsilon_1}(\tau)$ throughout $\tau \in [0,\Theta]$.
As previously established, the family $y_\epsilon(\tau)$ demonstrates both equicontinuity and uniform boundedness. Consequently, the Arzela-Ascoli theorem guarantees the existence of a decreasing sequence ϵ_k with $\epsilon_k \to 0$ as $k \to \infty$, such that $\lim_{k\to\infty} y_{\epsilon_k}(\tau)$ converges uniformly on $[0,\Theta]$. Denoting this uniform limit by $q(\tau)$, and observing the continuity of $f_i(\sigma, y_\epsilon(\sigma)), i = 1,2$ in the third argument, we obtain:

$$f_i\big(\sigma, y_{\epsilon_k}(\sigma)\big) \to f_i(\sigma, y(\sigma)) \text{ as } k \to \infty, i = 1,2.$$

Thus,

$$q(\tau) = \lim_{k\to\infty} y_{\epsilon_k}(\tau) = b(\tau) + \int_0^\tau H_1(\tau,\sigma) f_1(\sigma, q(\sigma)) d\sigma \int_0^\tau H_2(\tau,\sigma) f_2(\sigma, q(\sigma)) d\sigma$$

verifying that $q(\tau)$ constitutes a solution of the original integral equation. To establish the maximality property, let $y(\tau)$ represent an arbitrary solution of the integral equation:

$$y(\tau) = b(\tau) + \int_0^\tau H_1(\tau,\sigma) f_1(\sigma, y(\sigma)) d\sigma \int_0^\tau H_2(\tau,\sigma) f_2(\sigma, y(\sigma)) d\sigma$$

Application of Lemma 4 to $y(\tau)$ and $y_\epsilon(\tau)$ yields $y(\tau) < y_\epsilon(\tau)$ for $\tau \in [0,\Theta]$. Since $y_\epsilon(\tau)$ converges uniformly to $q(\tau)$ as $\epsilon \to 0$, it follows that $y(\tau) \le q(\tau)$ throughout $[0,\Theta]$. The uniqueness property of the maximal solution, as established in [9], confirms that $q(\tau)$ represents the maximal solution.
For the existence of the minimal solution, we employ a careful construction to ensure positivity. Consider the modified nonlinearities defined by:

$$f_{i\epsilon}(\sigma, y_\epsilon(\sigma)) = \max\left\{f_i(\sigma, y_\epsilon(\sigma)) - \epsilon, \frac{\epsilon}{2(1 + \|H_1\|\|H_2\|\Theta^2)}\right\}, i = 1,2.$$

This modification ensures that $f_{i\epsilon}(\sigma, y_\epsilon(\sigma)) > 0$ while maintaining the L^1-Carathéodory properties. The additional term guarantees that the product of integrals remains bounded away from zero, preserving the positivity of solutions.
Let $y_\epsilon(\tau)$ denote the corresponding solution. For parameters ϵ_1, ϵ_2 with $0 < \epsilon_2 < \epsilon_1$, we now have $y_{\epsilon_2}(\tau) > y_{\epsilon_1}(\tau)$ throughout $\tau \in [0,\Theta]$, establishing a decreasing family of solutions. The Arzela-Ascoli theorem again guarantees the existence of an increasing sequence $\epsilon_k \to 0$ such that $y_{\epsilon_k}(\tau)$ converges uniformly to a limit function $p(\tau)$. The uniform convergence and continuity properties ensure that:

$$p(\tau) = \lim_{k\to\infty} y_{\epsilon_k}(\tau) = b(\tau) + \int_0^\tau H_1(\tau,\sigma) f_1(\sigma, p(\sigma)) d\sigma \int_0^\tau H_2(\tau,\sigma) f_2(\sigma, p(\sigma)) d\sigma$$

verifying that $p(\tau)$ constitutes a solution. The minimality property follows by analogous reasoning to the maximal case, confirming that $p(\tau)$ represents the minimal solution. Assuming the functions f_1 and f_2 exhibit non-decreasing behavior in their second argument and the conditions of Corollary 3 remain satisfied, then the integral equation admits both maximal and minimal solutions.

Research Contributions and Comparative Analysis
Our investigation makes several distinct contributions to the mathematical theory of nonlinear integral equations. The primary innovation lies in developing a comprehensive analytical framework for establishing existence criteria for positive continuous solutions to Hammerstein-type quadratic integral equations. This methodological approach extends beyond prior research by incorporating more generalized kernel structures and nonlinearities, thereby expanding the applicable domain of existence theory for this important class of functional equations.
When situated within the broader scholarly conversation, our work demonstrates meaningful connections to several related research trajectories while maintaining its distinctive contributions. The approach adopted by [10], for instance, focused primarily on integral equations with simplified kernel structures, whereas our work incorporates more general Hammerstein-type operators with dual kernel components. Similarly, while [11]

employed Banach fixed-point methodology for Urysohn quadratic equations, we leverage Schauder's fixed-point theorem to establish existence under less restrictive compactness conditions.

The relationship between our current investigation and the research program developed by Fayed and collaborators deserves particular attention. Their extensive work on periodic solutions of neutral differential equations with various delay structures [12,13] establishes an important conceptual backdrop for understanding solution behavior in complex functional equations. Where their analysis emphasized periodic solutions in neutral differential systems, our investigation extends this line of inquiry to non-periodic solutions of Hammerstein-type quadratic integral equations, thereby addressing a fundamentally different class of problems that arise in distinct applied contexts.

Building on this foundation, the stability analysis in nonlinear neutral systems conducted by Althubiti and colleagues [14] provides crucial theoretical background that informs our understanding of solution behavior in sophisticated nonlinear systems. The methodological insights gained from Makhzoum's existence and uniqueness analysis for periodic solutions in nonlinear neutral differential equations [15] further strengthen our analytical framework, even as we adapt these approaches to the different challenges presented by quadratic integral equations.

Furthermore, recent contributions by [16] on Urysohn quadratic integral equations and the investigations of [12] [12 on nonlinear Langevin equations provide additional context for understanding the broader landscape of fixed-point applications in nonlinear analysis. The work of [17] on fixed point theorems in ordered b-metric spaces offers complementary methodological perspectives that enrich our theoretical approach.

What distinguishes our current contribution is the systematic development of extremal solution theory under monotonicity conditions, which represents a substantial theoretical advancement beyond standard existence results. By integrating insights from these related research programs while introducing novel analytical techniques specifically tailored to quadratic integral equations, we bridge an important gap in the mathematical literature. This integrated approach allows us to provide a more comprehensive theoretical framework for analyzing this important class of nonlinear functional equations, one that acknowledges its connections to broader mathematical traditions while advancing specific new results within its specialized domain.

Conclusion

This investigation has established comprehensive existence criteria for positive continuous solutions of Hammerstein-type quadratic integral equations through methodological application of Schauder's fixed-point theorem. Our analytical framework demonstrates that under appropriately formulated continuity, boundedness, and Carathéodory conditions, such nonlinear integral equations necessarily admit positive continuous solutions.

Furthermore, we have extended the theoretical landscape by establishing conditions guaranteeing the existence of maximal and minimal solutions under monotonicity constraints on the nonlinear components. These findings significantly expand the mathematical understanding of solution behavior for this important class of nonlinear functional equations and provide robust analytical tools for investigating similar mathematical structures in applied contexts.

The methodological approaches we have developed offer promising avenues for future research, including potential extensions to fractional-order quadratic integral equations, systems of coupled quadratic integral equations, and applications to boundary value problems with nonlinear integral constraints. Our theoretical framework provides a solid foundation for further analytical developments in the general theory of nonlinear integral equations.

Acknowledgements

The authors express profound gratitude to the anonymous reviewers for their meticulous examination of the manuscript, invaluable constructive criticism, and insightful scholarly suggestions that substantially enhanced the quality and rigor of this research presentation.

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