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A Novel Approach of Elicitation Procedure of The Generalized Trapezoidal Uniform Parameters by IVBM-Algorithm

Salma Omar Bleed *
[1] Department of Statistics, College of Science, Al-Asmarya Islamic University, Zliten, Libya
*Corresponding author: S.Ali@asmarya.edu.ly

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Abstract:
A simple elicitation procedure has been developed for the 6-parameter generalized trapezoidal uniform and trapezoidal distributions, assuming that the modal range needs to be estimated. Based on this procedure, a novel algorithm, called the Initial-Values-Bounded-Maximum (IVBM) algorithm, is proposed to estimate the unknown parameters of the 6-parameter generalized trapezoidal uniform and trapezoidal distributions. The algorithm is summarized in three steps, demonstrating how to estimate the parameters of distributions composed of three parts. It can also be applied to simpler distributions, such as triangular or uniform distributions. The significance of this work lies in introducing an efficient and flexible algorithm for parameter estimation in complex distributions, producing accurate and stable results that enhance the reliability of statistical and predictive modeling across various fields. A practical example is provided to illustrate the application of the IVBM algorithm to generalized trapezoidal uniform and trapezoidal distributions. The results indicate a consistent decrease in mean square error (MSE) as the number of iterations increases, highlighting the algorithm's effectiveness. Additionally, its adaptability allows it to improve the accuracy of models in diverse applications, enhancing both its theoretical and practical value.
Keywords: Elicitation procedure, generalized trapezoidal distribution, Generalized trapezoidal uniform distribution, Algorithm, ML method, Relative likelihood.

نهج جديد لإجراء استنباط معالم توزيع شبه المنحرف المنتظم المعمم باستخدام خوارزمية IVBM

سالمة عمر بالعيد علي*
قسم الإحصاء، كلية العلوم، الجامعة الأسمرية الإسلامية، زليتن، ليبيا

الملخص
طُوِّر إجراء استنباط بسيط للتوزيعات شبه المنحرفة المنتظمة ذات الستة معالم، والتوزيعات شبه المنحرفة المعممة لتقدير المعالم. وبناءً على هذا الإجراء، تُقترَح خوارزمية جديدة تُسمى خوارزمية القيم الابتدائية المحدودة بالحد الأقصى(IVBM) لتقدير المعالم المجهولة للتوزيعات شبه المنحرفة المعممة ذات الستة معالم. تُلخَّص الخوارزمية في ثلاث خطوات، توضح كيفية تقدير معالم التوزيعات المكونة من ثلاثة أجزاء. كما يمكن تطبيقها على توزيعات أبسط، مثل التوزيعات المثلثية أو المنتظمة. وتكمن أهمية هذا العمل في تقديم خوارزمية فعّالة ومرنة لتقدير معالم التوزيعات المعقدة، مما يُنتج نتائج دقيقة ومستقرة تُعزز موثوقية النمذجة الإحصائية والتنبؤية في مختلف المجالات. تم تقديم مثال عملي لتوضيح تطبيق خوارزمية(IVBM) على التوزيعات شبه المنحرفة المعممة والمنتظمة والتوزيعات شبه المنحرفة، وقد أشارت النتائج إلى انخفاض ثابت في متوسط مربع الخطأ (MSE) مع زيادة عدد التكرارات، مما يُبرز فعالية الخوارزمية، بالإضافة إلى، تُمكّنها قابلية التكيف من تحسين دقة النماذج في تطبيقات متنوعة، مما يُعزز قيمتها النظرية والعملية.

الكلمات المفتاحية: إجراء الاستنباط، توزيع شبه المنحرف المعمم، توزيع شبه المنحرف المعمم المنتظم، الخوارزمية، طريقة الامكان الاعظم، طريقة الامكان الاعظم النسبية.

Introduction

The proposed Initial-Values-Bounded-Maximum (IVBM) algorithm is developed to efficiently estimate the unknown parameters of generalized trapezoidal distributions. The algorithm operates by initiating a set of initial values and iteratively searching within predefined bounds to identify the maximum likelihood estimates. Its flexibility allows application not only to generalized trapezoidal distributions but also to other piecewise distributions, such as triangular or uniform distributions. Some authors have treated the study of the trapezoidal distribution. [1] proposed the trapezoidal distribution and specification of the modal value through means of a range of values rather than an unmarried point estimate. [2] and [3] formally describe the 6-parameters generalized trapezoidal distribution, representing the parameters $(a, b, c, d, M, N, \alpha)$ are distinctive with the aid of the arguments minimal, decrease mode, upper mode, maximum, increase rate, decay rate, and boundary ratio, respectively. [4] investigated the effective Measyrand estimators for symmetrical Trapezoidal probability distribution samples using Monte-Carlo method of simulation. [5] proposed a new distribution derived from the normal distribution by applying the trapezoidal rule, which is referred to as the trapezoidal normal distribution. In this study, we demonstrate the parameter estimation of the trapezoidal normal distribution using the standard Differential Evolution (DE) algorithm for sample sizes ranging from 10 to 100. The results indicate that, for all sample sizes, the K-S statistic of the trapezoidal normal distribution is consistently smaller than that of the standard normal distribution, suggesting a better fit. [6] derived the trapezoidal beta distribution and studied its properties. It compared to the standard parameter beta distribution The parameters are estimated using the expectation–maximization (EM) algorithm.

The probability density function (pdf) of the Generalized Trapezoidal (GT) distribution with 7-parameters were provided by [2] as:

$$f(t|\varsigma) = H \begin{cases} \alpha\left(\dfrac{t-a}{b-a}\right)^{M-1} & ; a \le t \le b \\ (1-\alpha)\left(\dfrac{t-b}{c-b}\right)+\alpha & ; b \le t \le c \\ \left(\dfrac{d-t}{d-c}\right)^{N-1} & ; c \le t \le d \end{cases} \qquad (1)$$

Where $H = \dfrac{2MN}{2\alpha(b-a)N + (\alpha+1)(c-b)MN + 2(d-c)M}$, $a < b < c < d > 0$.

Two novel elicitation procedures when $\alpha = 1$ and referring to it as Generalized Trapezoidal Uniform (GTU) distribution were developed by [3]. They assumed the parameters a and d are known, the modal range $[b, c]$ has been directly elicited from a substantive expert and suggested eliciting the relative likelihood $\frac{A_2}{A_1}$, $\frac{A_2}{A_3}$ and directly solve for the tail parameters M and N as:

$$M = \frac{A_2\,(b-a)}{A_1\,(c-b)}; \quad N = \frac{A_2\,(d-c)}{A_3\,(c-b)}.$$

Now, in this paper, assuming, the modal range $[b, c]$ and the unknown parameters of the GTU and trapezoidal distributions will be estimated using the proposed algorithm "**IVBM-algorithm**" with the relative likelihood $\frac{A_2}{A_1}$ and $\frac{A_2}{A_3}$. The Initial-Values-Bounded-Maximum IVBM-algorithm is a relevant technique for estimating the parameters of distributions, along with two, three parts or extra, including triangle, trapezoidal distributions. The steps of IVBM-algorithm are summarized as follows:

- Assume initial values for the limits of the variable t, and then obtain the initial values for M & N
- Estimate A's, and then obtain the estimates of c and b.
- Finally, obtain the maximum likelihood estimators for M and N.

Methods

Indirect Elicitation of GTU Parameters

The Generalized Trapezoidal Uniform (GTU) distribution is referred by setting $\alpha = 1$ in pdf.(1), then the pdf of GTU well be as follows:

$$f(t|\varsigma)=H\begin{cases}\left(\frac{t-a}{b-a}\right)^{M-1} & ;a\le t\le b\\ 1 & ;b\le t\le c\\ \left(\frac{d-t}{d-c}\right)^{N-1} & ;c\le t\le d\end{cases}\qquad(2)$$

Where $\varsigma=(a,b,c,d)$ and $H=\frac{MN}{N(b-a)+MN(c-b)+M(d-c)}$, $a<b<c<d>0$ .The mixture expressing of the pdf. (2) involving three densities with mixture weights

$$A_1=\frac{H(b-a)}{M},\quad A_2=H(c\text{-}b),\quad A_3=\frac{H(d-c)}{N}\ \text{ is:}$$

Such that:

$$f(t|\varsigma)=\sum_{i=1}^{3}A_i\, f_i(t|\varsigma)\ \text{with}\ \sum_{i=1}^{3}A_i=1$$

and

$$\begin{aligned}f_1(t|\varsigma)&=f_1(t|a,c)=\left(\frac{M}{b-a}\right)\left(\frac{t-a}{b-a}\right)^{M-1} && ;a\le t<b\\ f_2(t|\varsigma)&=f_2(t|c,d)=\frac{1}{(c\text{-}b)} && ;b\le t\le c\qquad(3)\\ f_3(t|\varsigma)&=f_3(t|d,b)=\left(\frac{N}{d-c}\right)\left(\frac{d-t}{d-c}\right)^{N-1} && ;c\le t<d\end{aligned}$$

Now, the steps of **IVBM-algorithm** of indirect elicitation for GTU parameters are summarized as follows:

IV - Step: Obtaining the Initial values

Let $t_1, t_2, ..., t_n$ a iid random sample of size n with order statistics be $t_{(1)}<t_{(2)}<...<t_{(n)}$. Assuming the initial values $d_0=b_0=t_{(n)}, a_0=c_0=t_{(1)}$. Since $f_1(x|\varsigma)$, $f_2(x|\varsigma)$, and $f_3(x|\varsigma)$ are pdf's in eq.(3), then the likelihood function for $f_1(x|\varsigma)$ and $f_3(x|\varsigma)$ are by definition:

$$L_1(\varsigma|t)=\prod_{i=1}^{n}f_1(t_i|a,b,M)=\prod_{i=1}^{n}\left(\frac{M}{b-a}\right)\left(\frac{t_i-a}{b-a}\right)^{M-1}\quad ;a\le t<b$$

$$L_3(\varsigma|t)=\prod_{i=1}^{n}f_3(t_i|c,d,N)=\prod_{i=1}^{n}\left(\frac{N}{d-c}\right)\left(\frac{d-t_i}{d-c}\right)^{N-1}\quad ;c\le t<d$$

with log likelihood function

$$LL_1(\varsigma|t)=n\log M-n\log(b-a)+(M\text{-}1)\sum_{i=1}^{n}\log\left(\frac{t_i-a}{b-a}\right)\quad ;a\le t<b$$

$$LL_3(\varsigma|t)=n\log N-n\log(d-c)+(N\text{-}1)\sum_{i=1}^{n}\log\left(\frac{d-t_i}{d-c}\right)\quad ;c\le t<d$$

By maximizing $LL_1(\varsigma|t)$ and $LL_3(\varsigma|t)$ with respect to M, N respectively, we get

$$M_0=n\left[n\ln(b-a)-\sum_{i=1}^{n}\ln(t_i-a)\right]^{-1}\qquad(4)$$

$$N_0=n\left[n\ln(d-c)-\sum_{i=1}^{n}\ln(d-t_i)\right]^{-1}\qquad(5)$$

The pdf.(3) may be expressed as a mixture (see, e.g., van Dorp and Kotz (2003)), such that $f(t|\varsigma)=\sum_{i=1}^{3}A_i\, f_i(t|\varsigma)$, then the mixture weights $A_{0i},\ i=1,2,3$ with $\sum_{i=1}^{3}A_{0i}=1$ can be derived as follows:

$$A_{01}=\frac{H(b-a)}{M},\qquad A_{02}=H(c\text{-}b),\ A_{03}=\frac{H(d-c)}{N}\qquad(6)$$

B - Step: Relative likelihood estimation of the bounded parameters

Suggesting to indirect elicitation of the parameters b and c from the relative likelihood $\frac{A_2}{A_1}$ and $\frac{A_2}{A_3}$, utilizing relationships:

$$\frac{A_2}{A_1}=\frac{\hat{M}_0\,(c-b)}{(b-a)}\rightarrow\ c=\frac{(\hat{M}_0+\theta)b-\theta a}{\hat{M}_0}\ ,\theta=\frac{A_2}{A_1} \tag{7}$$

$$\frac{A_2}{A_3}=\frac{\hat{N}_0\,(c-b)}{(d-c)}\rightarrow c=\frac{\lambda d+\hat{N}_0 b}{\hat{N}_0+\lambda}\ ,\lambda=\frac{A_2}{A_3} \tag{8}$$

Equalization (7) and (8), the estimate value of b is:

$$\hat{b}=\frac{\lambda\hat{M}_0 d+\theta a\left[\hat{N}_0+\lambda\right]}{\hat{N}_0\theta+\lambda\left[\hat{M}_0+\theta\right]} \tag{9}$$

Substituting the value of (9) in (7) or (8), the estimate value of c is gotten.

M - Step: Maximum Likelihood Estimators

The pdf. (2) can be rewriting including the mixture weights as follows:

$$f_1^*(t|\varsigma)=f_1(t|a,b)=\left(\frac{MA_1}{b-a}\right)\left(\frac{t-a}{b-a}\right)^{M-1} \qquad ;a\le t<b$$

$$f_2^*(t|\varsigma)=f_2(t|b,c)=\frac{A_2}{(c-b)} \qquad ;b\le t\le c$$

$$f_3^*(t|\varsigma)=f_3(t|c,d)=\left(\frac{NA_3}{d-c}\right)\left(\frac{d-t}{d-c}\right)^{N-1} \qquad ;c\le t<d$$

Therefore,

$$f(t|\varsigma)=\sum_{i=1}^{3}A_i\,f_i(t|\varsigma)=\left(\frac{MA_1}{b-a}\right)\left(\frac{t-a}{b-a}\right)^{M-1}+\frac{A_2}{(c-b)}+\left(\frac{NA_3}{d-c}\right)\left(\frac{d-t}{d-c}\right)^{N-1} \tag{10}$$

Solving the following 2-normal equations numerically to get $\hat{M}$ and $\hat{N}$.

$$\frac{\partial LL(\varsigma|t)}{\partial M}=\frac{n}{M}-\frac{n[KN+Q]}{GN+KMN+QM}+\sum_{i=1}^{n}\frac{T_i^{M-1}\ln T_i}{\left[T_i^{M-1}-U_i+1\right]+U_i+W_i^{N-1}}$$

$$\frac{\partial LL(\varsigma|t)}{\partial N}=\frac{n}{N}-\frac{n[G+KM]}{GN+KMN+QM}+\sum_{i=1}^{n}\frac{W_i^{N-1}\ln W_i}{\left[T_i^{M-1}-U_i+1\right]+U_i+W_i^{N-1}} \tag{11}$$

Where

$$T_i=\frac{t_i-a}{b-a},U_i=\left(\frac{t_i-b}{c-b}\right),W_i=\left(\frac{d-t_i}{d-c}\right);G=(b-a),K=(c-b),Q=(d-c).$$

2nd iteration:

1. the ML estimated values $(\hat{M},\hat{N})$ and $(\hat{b},\hat{c})$ of the 1st iteration will be used as the initial values for computing the mixture weights $\hat{A}_i,\ i=1,2,3$ such that $\hat{A}_{01}=\frac{H(\hat{b}-a)}{\hat{M}}$, $\hat{A}_{02}=H(\hat{c}-\hat{b})$, $.\sum_{i=1}^{3}\hat{A}_{0i}=1$, with $\hat{A}_{03}=\frac{H(d-\hat{c})}{\hat{N}}$
2. The estimated values of $(\hat{M},\hat{N})$, $(\hat{b},\hat{c})$ and $\hat{A}_i,\ i=1,2,3$ will be used to get the ML estimated values $\left(\hat{\hat{M}},\hat{\hat{N}}\right)$.
3. Using $\left(\hat{\hat{M}},\hat{\hat{N}}\right)$ and $\hat{A}_i,\ i=1,2,3$ to compute $\left(\hat{\hat{b}},\hat{\hat{c}}\right)$ as in eqs. (6, 7, 8).

mth iteration:

1. the ML estimated values $\zeta^{(m-1)} = \left(M^{(m-1)}, N^{(m-1)}\right)$ and $\left(b^{(m-1)}, c^{(m-1)}\right)$ of the last iteration ((m-1)th iteration) will be used as the initial values of the current iteration (mth iteration) for computing the mixture weights $A^{(m)}{}_i,\ i=1,2,3$ such that $A^{(m)}{}_{01} = \frac{H(b-a)}{M}$, $A^{(m)}{}_{02} = H(c-b)$, $.\sum_{i=1}^{3} A_{0i} = 1$, with $A^{(m)}{}_{03} = \frac{H(d-c)}{N}$
2. The estimated values of $\zeta^{(m-1)} = \left(M^{(m-1)}, N^{(m-1)}\right)$, $\left(b^{(m-1)}, c^{(m-1)}\right)$ and $A^{(m)}{}_i,\ i=1,2,3$ will be used to get the ML estimated values $\zeta^{(m)} = \left(M^{(m)}, N^{(m)}\right)$ of the current iteration (mth iteration).
3. Using $\zeta^{(m)} = \left(M^{(m)}, N^{(m)}\right)$ and $A^{(m)}{}_i,\ i=1,2,3$ to compute $\left(b^{(m)}, c^{(m)}\right)$ as in eqs. (6, 7, 8).

Therefore, the new parameters values $\zeta^{(m+1)} = \left(M^{(m+1)}, N^{(m+1)}\right)$ and $\left(b^{(m+1)}, c^{(m+1)}\right)$ are obtained at the iteration (m) and the iterations continued until the values of A's convergence, stability and $|\theta^{(m+1)} - \theta^{(m)}| < \varepsilon\ , \varepsilon > 0$ where ε a very small amount.

Indirect Elicitation of Trapezoidal Parameters

The trapezoidal distribution is referred by setting $\alpha = 1$ and $M = N = 2$ in pdf. (1). Then the pdf of the trapezoidal distribution well be as follows:

$$f(t|\varsigma) = H \begin{cases} \left(\frac{t-a}{b-a}\right) & ;a \le t \le b \\ 1 & ;b \le t \le c \\ \left(\frac{d-t}{d-c}\right) & ;c \le t \le d \end{cases} \qquad (12)$$

Where $\varsigma = (a,b,c,d)$ and $H = 2[(d-a)+(c-b)]^{-1}$, $a < b < c < d > 0$.

The mixture expressing of the pdf. (2) involving three densities with mixture weights $A_1 = \frac{H(b-a)}{2}$, $A_2 = H(c-b)$, $A_3 = \frac{H(d-c)}{2}$ is:

$$f(t|\varsigma) = \sum_{i=1}^{3} A_i\ f_i(t|\varsigma) \text{ with } \sum_{i=1}^{3} A_i = 1$$

with

$$f_1(t|\varsigma) = f_1(t|a,c) = \left(\frac{2}{b-a}\right)\left(\frac{t-a}{b-a}\right) \quad ;a \le t < b$$

$$f_2(t|\varsigma) = f_2(t|c,d) = \frac{1}{(c-b)} \quad ;b \le t \le c$$

$$f_3(t|\varsigma) = f_3(t|d,b) = \left(\frac{2}{d-c}\right)\left(\frac{d-t}{d-c}\right) \quad ;c \le t < d$$

IV - Step: Obtaining the Initial values

In this step, the Initial values of the mixture weights $A_i,\ i=1,2,3$ are obtained as follows:

1. Let $t_1, t_2, \ldots, t_n$ a iid random sample of size n with order statistics be $t_{(1)} < t_{(2)} < \ldots < t_{(n)}$ then $t_{(1)} \equiv \hat{a}$, $t_{(n)} \equiv \hat{d}$ and assume that $\hat{a} < b < c,\ b < c < \hat{d}$.
2. Computing the Initial values of the mixture weights $A_i,\ i=1,2,3$ as follows:

The pdf. (12) may be expressed as a mixture, such that $f(t|\varsigma) = \sum_{i=1}^{3} A_i\ f_i(t|\varsigma)$ then the mixture weights $A_{0i},\ i=1,2,3$ with $\sum_{i=1}^{3} A_{0i} = 1$ are

$$A_{01} = \frac{H(b-a)}{2}, \quad A_{02} = H(c-b),\ A_{03} = \frac{H(d-c)}{2} \qquad (13)$$

B - Step: Relative likelihood estimation of the bounded parameters

In this step, the relative likelihood $\frac{A_2}{A_1}$ and $\frac{A_2}{A_3}$ with (b_0, c_0) and $A_{0i},\ i=1,2,3$ are used to estimate the parameters $[b, c]$, as follows:

1. suggesting to indirect elicitation of the parameters b and c from the relative likelihood $\frac{A_2}{A_1}$ and $\frac{A_2}{A_3}$, utilizing relationships:

$$\frac{A_2}{A_1} = \frac{2(c-b)}{(b-a)} \rightarrow c = \frac{(2+\theta)b - \theta a}{2}, \theta = \frac{A_2}{A_1} \quad (14)$$

$$\frac{A_2}{A_3} = \frac{2(c-b)}{(d-c)} \rightarrow c = \frac{\lambda d + 2b}{2+\lambda}, \lambda = \frac{A_2}{A_3} \quad (15)$$

1. Equalization (27) and (28), the estimate value of b is:

$$\hat{b} = \frac{2\lambda d + \theta a\ [2+\lambda]}{2\theta + \lambda[2+\theta]} \quad (16)$$

Substituting the value of (16) in (14) or (15), the estimate value of c is gotten

2nd iteration:

1. The estimated values $(\hat{b}, \hat{c})$ of the 1st iteration will be used as the initial values for computing the mixture weights $\hat{A}_i,\ i=1,2,3$ such that $\hat{A}_{01} = \frac{H(\hat{b}-a)}{2}$, $\hat{A}_{02} = H(\hat{c}-\hat{b})$, , with $\hat{A}_{03} = \frac{H(d-\hat{c})}{2}$. $\sum_{i=1}^{3} \hat{A}_{0i} = 1$
2. The estimated values of $\hat{A}_i,\ i=1,2,3$ will be used to compute $\left(\hat{\hat{b}}, \hat{\hat{c}}\right)$ as in eqs. (6, 7, 8).

mth iteration:

1. The estimated values $\left(b^{(m-1)}, c^{(m-1)}\right)$ of the last iteration ((m-1)th iteration) will be used as the initial values of the current iteration (mth iteration) for computing the mixture weights $A^{(m)}{}_i,\ i=1,2,3$ such that $A^{(m)}{}_{01} = \frac{H(b-a)}{M}$, $A^{(m)}{}_{02} = H(c-b)$, . $\sum_{i=1}^{3} A_{0i} = 1$, with $A^{(m)}{}_{03} = \frac{H(d-c)}{N}$
2. The estimated values of $A^{(m)}{}_i,\ i=1,2,3$ will be used to compute $\left(b^{(m)}, c^{(m)}\right)$ as in eqs. (6, 7, 8).

Therefore, the new parameters values $\left(b^{(m+1)}, c^{(m+1)}\right)$ are obtained at the iteration (m), and the iterations continued until the values of A's convergence, stability and $|\theta^{(m+1)} - \theta^{(m)}| < \varepsilon\ , \varepsilon > 0$, where ε a very small amount.

An illustrative Example

This section presents the IVBM-algorithm to compute the exact values of the initial values problem to estimate the unknown parameters and show the practical application of the IVBM-algorithm under the GTU and Trapezoidal distributions. The order statistics of the data in Johnson and Kotz (1999) is used and is: 0.1, 0.25, 0.3, 0.4, 0.45, 0.6, 0.75, 0.8. The numerical solution for obtaining the initial values and the MLE and their properties is performed according to the following steps:

Case I: GTU Parameters Estimation when (A'S known)

I - Step: In this step, the Initial values of M_0, N_0 and of the mixture weights $A_i,\ i=1,2,3$ are obtained as follows:

1. Let $t_1, t_2, \ldots, t_n$ a iid random sample of size n with order statistics be $t_{(1)} < t_{(2)} < \ldots < t_{(n)}$ then $\hat{a} = 0.09$, $\hat{d} = 0.801$.
2. Assuming the initial values $(b_0 = 0.09, c_0 = 0.801)$, and then obtaining the Initial values of M_0, N_0 according to eq. (4), eq. (5).
3. Computing the Initial values of the mixture weights $A_i,\ i=1,2,3$ according to eq. (6).

4. Using (M_0, N_0) and $\hat{A}_i,\ i = 1,2,3$ the MLE of (M, N) are estimated by solving the normal equations (11) numerically.

The result was very satisfactory, after the 4th iteration, very small MSE with stability of the estimated results are gotten. That values were used at the i^{th} iteration and used as initial values in the $(i + 1)^{th}$ iteration. It noted that MSE decreases as the number of iterations increases. Thus, we repeat the same steps; continuing the iterations until the values of A's convergence or be similar, see the results in Table (1). The distribution of GTU can be represented by three stages (formation-growth-sustained growth and stability) as shown in Figure 1 and Figure 2.

Table 1. GTU Parameters Estimation

$a_0 = 0.09,\ b_0 = 0.3,\ c_0 = 0.6,\ d_0 = 0.801$							
Iteration	Obtaining Init. values		Est. of mixture weights			MLE	MSE
	M	0.8995	$\hat{A}_1$	0.28	$\hat{M}$	0.9468	0.002200000
1	N	0.6471	$\hat{A}_2$	0.36	$\hat{N}$	0.6591	0.000100000
	-	-	$\hat{A}_3$	0.37	-	-	-
	M	0.9468	$\hat{A}_1$	0.30	$\hat{M}$	0.9468	0.0000000002
2	N	0.6591	$\hat{A}_2$	0.36	$\hat{N}$	0.6591	0.0000000007
	-		$\hat{A}_3$	0.34	-	-	-
	M	0.9468	$\hat{A}_1$	0.27	$\hat{M}$	0.8456	0.0000000002
3	N	0.6591	$\hat{A}_2$	0.36	$\hat{N}$	0.7163	0.0000000007
	-	-	$\hat{A}_3$	0.37	-	-	-

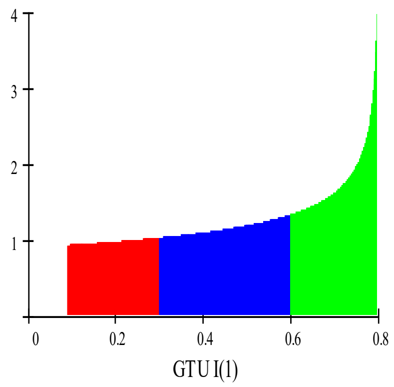

Figure 1. The 3-stages of GTU distribution at (1th iteration)

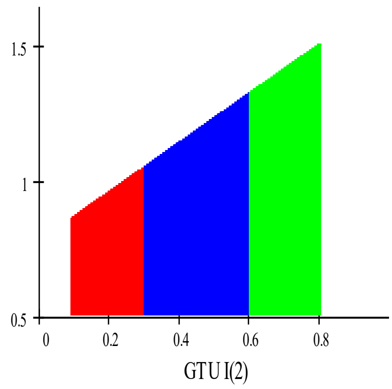

Figure 2: The 3-stages of GTU distribution at (2nd iteration)

Case II: GTU Parameters Estimation when (A'S Unknown)

IV - Step: Obtaining the Initial values

In this step, the Initial values of M_0, N_0 and of the mixture weights $A_i,\ i = 1,2,3$ are obtained as follows:

1. Let $t_1, t_2, \ldots, t_n$ a iid random sample of size n with order statistics be $t_{(1)} < t_{(2)} < \ldots < t_{(n)}$ then $\hat{a} = 0.09$, $\hat{d} = 0.801$.
2. Assuming the initial values $(b_0 = 0.09, c_0 = 0.801)$, and then obtaining the Initial values of M_0, N_0 according to eq. (3), eq. (4).
3. Computing the Initial values of the mixture weights $A_i,\ i = 1,2,3$ according to eq. (6).
4. Using (M_0, N_0) and $\hat{A}_i,\ i = 1,2,3$ the MLE of (M, N) are estimated by solving the normal equations (11) numerically.
5. Using (b_0, c_0), MLE of $(\hat{M}, \hat{N})$ and $\hat{A}_i,\ i = 1,2,3$ the parameters $[b, c]$ are computed according to eq. (9).

Therefore, the iterations continued until the values of A's convergence, stability and $|\theta^{(m+1)} - \theta^{(m)}| < \varepsilon \ , \varepsilon > 0$, where ε a very small amount. Note that the MSE decreases as the number of iterations increases, the results indicates that, the values of $(a < b < c < d)$ as well as MSE's are small. Therefore, the proposed approach of computing the initial values is easy to use and may be relied upon in other topics. See the results in Table 2. The distribution of GTU can be represented by three stages (formation-growth-sustained growth and stability) as shown in Figure 3 and Figure 4.

Table 2. GTU Parameters Estimation when (A'S Unknown)

$a_0 = 0.09, \ b_0 = 0.3, \ c_0 = 0.6, \ d_0 = 0.801$										
Iteration	Init. values		Obtaining Init. values		Est. of mixture weights			MLE	MSE	Est. of elative L.L.E U sing last ests.	
	b	0.3	M	0.8995	$\hat{A}_1$	0.28	$\hat{M}$	0.9468	0.0022000000000	b	0.307
1	c	0.6	N	0.6471	$\hat{A}_2$	0.36	$\hat{N}$	0.6591	0.000100000000	c	0.600
	-	-	-	-	$\hat{A}_3$	0.37	-	-	-	-	-
	b	0.307	M	0.9468	$\hat{A}_1$	0.28	$\hat{M}$	0.9537	0.0000474000000	b	0.308
2	c	0.600	N	0.6591	$\hat{A}_2$	0.35	$\hat{N}$	0.6599	0.0000006000000	c	0.600
	-	-	-	-	$\hat{A}_3$	0.37	-	-	-	-	-
	b	0.308	M	0.9537	$\hat{A}_1$	0.28	$\hat{M}$	0.9546	0.000000864	b	0.308
3	c	0.600	N	0.6599	$\hat{A}_2$	0.35	$\hat{N}$	0.6600	0.00000001	c	0.600
	-	-	-	-	$\hat{A}_3$	0.37	-	-	-		-
	b	0.308	M	0.9546	$\hat{A}_1$	0.27	$\hat{M}$	0.9546	0.0000000008584	b	0.308
4	c	0.600	N	0.6600	$\hat{A}_2$	0.34	$\hat{N}$	0.6600	0.0000000000005	c	0.600
	-	-	-	-	$\hat{A}_3$	0.36	-	-	-	-	-

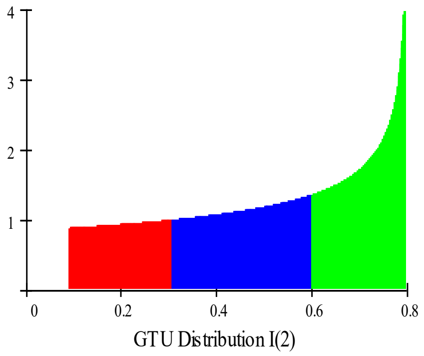

Figure 3. The 3-stages of GTU distribution at (2nd iteration)

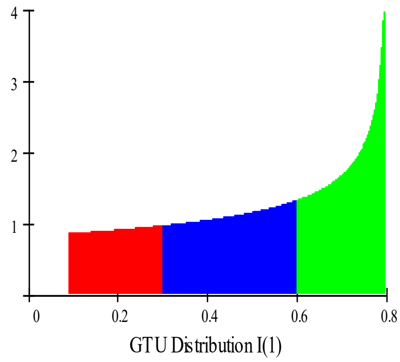

Figure 4. The 3-stages of GTU distribution at (1th iteration)

Discussion

In this paper, a new Initial-Values-Bounded-Maximum (IVBM) algorithm with upper bounds on initial values is proposed, along with a simple elicitation procedure to estimate the unknown parameters of a 6-parameter generalized trapezoidal uniform (GTU) distribution, including the parameters of the typical range. The significance of this work lies in developing an algorithm for estimating the parameters of a generalized distribution, providing a foundation for future research and further applications. Compared to traditional distributions, such as triangular or standard trapezoidal distributions, the 6-parameter GT distribution offers a higher degree of flexibility in representing various types of data. The proposed algorithm introduces a novel tool for solving the problem of parameter estimation in complex distributions that may be difficult to handle using conventional methods. From a practical standpoint, it is not limited to the generalized trapezoidal distribution but can also be applied to other piecewise distributions, such as triangular or uniform, making it versatile and widely applicable. Furthermore, the stable estimates and low mean square error (MSE) indicate that models based on this algorithm are more accurate and reliable for practical applications, including environmental, economic, and engineering data analysis. The observed decrease in MSE with increasing iterations demonstrates that the algorithm is not only efficient but also dependable in applications requiring repeated simulations or calculations. A practical example using a uniform or trapezoidal distribution further illustrates the algorithm's effectiveness and reinforces the credibility of this work.

Conclusion

Two novel elicitation procedures for the GTU distribution were proposed by [3], assuming that the modal range parameters were directly elicited from a substantive expert. In this paper, a new Initial-Values-Bounded-Maximum (IVBM) algorithm is proposed, along with a simple elicitation procedure to estimate the unknown parameters of the seven-parameter GT distribution, including the modal range parameters. The IVBM algorithm is summarized in three steps to demonstrate how to estimate the parameters of distributions composed of three parts. The first step, called the IV-step, uses mixture weights to obtain initial values of the unknown parameters within fixed lower and upper bounds. The second step, called the B-step, applies the relative likelihood to estimate the parameters based on the initial values and the fixed bounds. The third step, called the M-step, employs the maximum likelihood method to finalize the estimation of the distribution parameters. The IVBM algorithm is explained and applied under GT, GTU, and trapezoidal distributions. Practical implementation demonstrates stable and accurate parameter estimates with very low mean squared error (MSE), confirming the robustness and effectiveness of the IVBM algorithm. The results from practical applications on the generalized trapezoidal, uniform, and ordinary trapezoidal distributions showed highly satisfactory performance, with a significant reduction in MSE and stable estimates. These findings highlight the robustness of the algorithm and establish it as a valuable tool for both theoretical and applied research.

Disclaimer

The article has not been previously presented or published, and is not part of a thesis project.

Conflict of Interest

There are no financial, personal, or professional conflicts of interest to declare.

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