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Periodic Solutions for A Class of Integro-Differential Equations with Vriable Delay and Exponential Nonlinearity

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Abstract

This paper establishes the existence and uniqueness of a periodic solution for a nonlinear integro-differential equation with variable delay, employing Kwasniewski's Fixed-point Theorem. We consider the equation $x'(t) + a(t)x(t) + \int_{t-\tau(t)}^{t} K(t,s) \cdot x(s)e^{-|x(s)|}ds = 0$ where $a: R \to R$, $\tau: R \to R^+$ and $K: R \times R \to R$ are continuous, $\boldsymbol{\omega}$-periodic functions in t, with $\tau(t)$ satisfying appropriate regularity and boundedness conditions (e.g., $0 \leq \tau(t) \leq \tau_0 < \infty$). The nonlinear term $x(s)e^{-|x(s)|}$ exhibits a natural boundedness property: the function $f(x)= xe^{-|x|}$ is globally bounded on R, satisfying $| g(x) | \leq \frac{1}{e}$ for all $x \in R$. This crucial feature ensures the integral operator is well-behaved and facilitates the application of Kwasniewski's theorem within the Banach space $C_\omega$of continuous $\boldsymbol{\omega}$ -periodic functions. To apply the theorem, we decompose the associated integral operator into the sum of a compact operator and a contraction mapping, thereby verifying the requisite conditions. Uniqueness of the periodic solution is further established through refined differential estimates and a careful analysis of the Lipschitz properties induced by the exponential nonlinearity. The results presented herein constitute a significant generalization of prior work in the literature, as they address the challenging combination of variable delay and a highly nonlinear, non-Lipschitz kernel that cannot be handled by classical methods. This work contributes not only to the theoretical framework of functional differential equations but also provides a robust analytical tool applicable to models in population dynamics, neural networks, and control systems with memory effects.

Key words: Periodic solutions, Krasnoselskii fixed point theorem, Integro-differential equations, Variable delay, Uniqueness.

حلول دورية لفئة من المعادلات التفاضلية التكاملية ذات التأخير المتغير واللاخطية الاسية

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الملخص

يؤســـس هذا البحث وجود ووحدانية الحلول الدورية لفئة من المعادلات التكاملية التفاضـــلية غير الخطية التي تتميز بحدود التأخير المتغير. يستخدم إطارنا النظري مبرهنة النقطة الثابتة لكفاشنيفسكي كأداة تحليلية أساسية. نقوم بدراسة المعادلة التفاضلية:

$x'(t) + a(t)x(t) + \int_{t-\tau(t)}^{t} K(t,s) \cdot x(s)e^{-|x(s)|}ds = 0$ حيث $a: \mathbb{R} \to \mathbb{R}$، $\tau: \mathbb{R} \to \mathbb{R}^+$، و $K: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$تمثل دوال متصـــلة ودورية ω-بالنســـبة للمتغير الزمني t. تحقق دالة التأخير $\tau(t)$معايير صـــارمة للانتظام والمحدودية، وتحديداً. $0 \leq \tau(t) \leq \tau_0 < \infty$.
تكمن خاصية مميزة لنظامنا في الحد غير الخطي $x(s)e^{-|x(s)|}$، والذي يُظهر خصائص محدودية متأصلة. الدالة $f(x) = xe^{-|x|}$محدودة عالمياً على $\mathbb{R}$، وتحقق المتباينة $|f(x)| \leq 1/e$ لجميع $x \in \mathbb{R}$. هذه الخاصية الأساسية تضمن حسن تشكيل المؤثر التكاملي وتسهل التطبيق الصارم لمبرهنة كفاشنيفسكي داخل الفضاء البناخي C_ω للدوال المتصلة الدورية . ينطوي

منهجنا على التحليل الاستراتيجي للمؤثر التكاملي المرتبط إلى مجموع مؤثر مضغوط ومؤثر انقباضي، مما يحقق الشروط الضرورية لمبرهنة النقطة الثابتة. يتم إثبات وحدانية الحل الدوري بصرامة من خلال تقديرات تفاضلية متطورة وتحليل شامل لخصائص الاستمرارية الليبشيتزية المستحثة بواسطة اللاخطية الأسية.
تمثل المساهمات النظرية المقدمة في هذا العمل تقدماً جوهرياً عن الأدبيات الموجودة، حيث تعالج بنجاح التركيب المتحدي تحليلياً لآليات التأخير المتغير والدوال النواة عالية اللاخطية وغير الليبشيتزية التي لا يمكن التعامل معها باستخدام المنهجيات التقليدية. لا يثري هذا البحث الأسس النظرية للمعادلات التفاضلية الوظيفية فحسب، بل يوفر أيضاً إطاراً تحليلياً قوياً مع تطبيقات مهمة في نمذجة ديناميكيات السكان، وتحليل الشبكات العصبية، وأنظمة التحكم التي تتضمن تأثيرات الذاكرة.

الكلمات المفتاحية :المعادلات التكاملية التفاضلية، التأخير المتغير، الحلول الدورية، مبرهنة النقطة الثابتة لكفاشنيفسكي، التحليل غير الخطي، المعادلات التفاضلية الوظيفية

Introduction

The study of periodic solutions for differential equations with delays represents a fundamental area of mathematical analysis with significant applications across various scientific and engineering disciplines. These functional differential equations provide a natural framework for modeling phenomena in biology, economics, and physics where a system's future state depends on its historical behavior [1, 2, 3]. The field has been substantially influenced by foundational work from researchers [3, 4] and others who have investigated existence, uniqueness, and stability properties for various types of equations, including neutral systems [5, 6, 7, 8] and sublinear Duffing equations [9].

This paper addresses the existence and uniqueness of periodic solutions for the nonlinear integro-differential equation with variable delay:

$$x'(t) + a(t)x(t) + \int_{t-\tau(t)}^{t} K(t,s) \cdot x(s)e^{-|x(s)|}ds = 0 \tag{1}$$

where the functions $a(t), \tau(t)$, and $K(t,s)$ are periodic. This class of equations is particularly relevant as the integral term captures the cumulative effect of past states over a time-varying interval, a feature commonly observed in real-world systems. Previous research has examined integrodifferential equations with delays in various contexts, including studies of positive solutions with unbounded delay [9] and sub-exponential solutions [10].

The establishment of periodic solutions for such equations constitutes a classical problem typically addressed through fixed-point theory [11]. Several investigations have applied this methodology to neutral functional differential equations, as demonstrated in the works of [12,13,14]. While these studies successfully established existence and uniqueness results for their respective models, our current work addresses a distinct problem characterized by an exponential nonlinearity combined with variable delay.

The primary objective of this paper is to provide a rigorous proof for the existence and uniqueness of ω-periodic solutions for this equation. Our approach employs the Krasnoselskii Fixed Point Theorem, which effectively combines both contraction and compactness properties. The exponential nonlinearity $g(x) = xe^{-|x|}$ provides natural boundedness conditions essential for the application of fixed-point theory, while the variable delay $\tau(t)$ introduces additional complexity that reflects more realistic system behaviors.

The key contribution of this work lies in its methodological rigor and the derivation of sufficient conditions that guarantee the existence and uniqueness of periodic solutions. By leveraging the specific properties of the exponential nonlinearity, we establish precise bounds and Lipschitz constants that facilitate the application of fixed-point theory.

The remainder of this paper is organized as follows: Section 2 presents necessary definitions and preliminary results, Section 3 contains our main theorem with detailed proof, and Section 4 summarizes our findings and presents concluding remarks.

Preliminaries and Notation

Assumptions

The following assumptions are made on the functions in our equation:

1. The function $a(t)$ is continuous and ω-periodic, with $\int_0^{\omega} a(t)dt > 0$.
2. The delay function $\tau(t)$ is continuous and ω-periodic, with $0 \le \tau(t) \le \tau_0$ for some constant τ_0.
3. The function $K(t,s)$ is continuous and ω-periodic in t, i.e., $K(t+\omega, s+\omega) = K(t,s)$, and its maximum value is denoted as $\max_{t,s}|K(t,s)|$.

Definition 1. The space C_ω is the Banach space of continuous ω-periodic functions $x: \mathbb{R} \to \mathbb{R}$ with the norm:

$$\|x\| = \sup_{t\in[0,\omega]} |x(t)|.$$

Definition 2. The Green's function for the linear equation $x'(t) + a(t)x(t) = f(t)$ is defined as:

$$G(t,s) = \frac{e^{-\int_s^t a(u)du}}{1 - e^{-\int_0^\omega a(u)du}}$$

Remark 1. Under Assumption (1), the Green's function is well-defined. It satisfies $G(t+\omega, s+\omega) = G(t,s)$ and there exist constants $m, M > 0$ such that:

$$m \le |G(t,s)| \le M \quad \text{for all } t, s \in \mathbb{R}.$$

Theorem 1 (Krasnoselskii Fixed Point Theorem). Let M be a closed, convex, nonempty subset of a Banach space $(X, \|\cdot\|)$. Suppose that A and B map M into X such that:

1. $Ax + By \in M$ for all $x, y \in M$,
2. A is compact and continuous,
3. B is a contraction mapping.

Then there exists $z \in M$ with $z = Az + Bz$.

Main Results

We transform equation (1) into an equivalent integral equation. An ω periodic solution to (1) is a fixed point of the operator $T: C_\omega \to C_\omega$ defined by:

$$(Tx)(t) = -\int_{t-\omega}^{t} G(t,s)\left[\int_{s-\tau(s)}^{s} K(s,u)x(u)e^{-|x(u)|}du\right]ds \quad (2)$$

Lemma 2. For the function $g(x) = xe^{-|x|}$, we have:

1. $|g(x)| \le \frac{1}{e}$ for all $x \in \mathbb{R}$
2. $|g(x) - g(y)| \le |x - y|$ for all $x, y \in \mathbb{R}$

Proof. For $x \ge 0, g(x) = xe^{-x}$, and $g'(x) = (1-x)e^{-x}$. The critical point is at $x = 1$, where $g(1) = 1/e$. For $x < 0, g(x) = xe^{x}$, and $g'(x) = (1+x)e^{x}$. The critical point is at $x = -1$, where $g(-1) = -1/e$. Since g is an odd function, the maximum absolute value is $\sup_{x\in\mathbb{R}} |g(x)| = 1/e$.

For the Lipschitz property, we find the supremum of $|g'(x)|$:

- For $x > 0$: $|g'(x)| = |(1-x)e^{-x}| \le 1$(achieved at $x = 0$)
- For $x < 0$: $|g'(x)| = |(1+x)e^{x}| \le 1$ (limit as $x \to 0^-$)

Thus, $\sup_{x\in\mathbb{R}} |g'(x)| = 1$. By the Mean Value Theorem, $|g(x) - g(y)| \le |x - y|$.

Lemma 3. The operator $T: C_\omega \to C_\omega$ defined in (2) is well-defined and maps C_ω into itself.

Proof. Let $x \in C_\omega$. Define the inner integral:

$$I(s) = \int_{s-\tau(s)}^{s} K(s,u)x(u)e^{-|x(u)|}du$$

Since $K(s,u), \tau(s)$, and $x(u)e^{-|x(u)|}$ are continuous and the integration interval is compact, $I(s)$ is continuous. By periodicity of K, τ, and x, we have:

$I(s+\omega) = \int_{s+\omega-\tau(s+\omega)}^{s+\omega} K(s+\omega, u)x(u)e^{-|x(u)|}du = \int_{s-\tau(s)}^{s} K(s,v)x(v)e^{-|x(v)|}dv = I(s)$,

using the substitution $v = u - \omega$. Thus, $I(s)$ is ω-periodic.

Now, $(Tx)(t) = -\int_{t-\omega}^{t} G(t,s)I(s)ds$. Since $G(t,s)$ and $I(s)$ are continuous, Tx is continuous. The periodicity follows from:

$$(Tx)(t+\omega) = -\int_{t}^{t+\omega} G(t+\omega, s)I(s)ds = -\int_{t-\omega}^{t} G(t,u)I(u)du = (Tx)(t)$$

using the periodicity of G and I.

Lemma 4. The operator $A: C_\omega \to C_\omega$ defined by

$$(Ax)(t) = -\int_{t-\omega}^{t} G(t,s)\left[\int_{s-\tau(s)}^{s} K(s,u)x(u)e^{-|x(u)|}du\right]ds$$

is compact and continuous.

Proof. For continuity, let $x, y \in C_\omega$. Then:

$$\begin{aligned}\|Ax - Ay\| &= \sup_{t\in[0,\omega]} |(Ax)(t) - (Ay)(t)| \\ &\leq M \cdot \max_{t,s}|K(t,s)| \cdot \|x - y\| \cdot \omega \cdot \tau_0\end{aligned}$$

Thus, A is continuous.
For compactness, let $S \subset C_\omega$ be bounded. For $x \in S$, we have:

$$|(Ax)(t)| \leq M \cdot \max_{t,s}|K(t,s)| \cdot \frac{1}{e} \cdot \omega \cdot \tau_0$$

so $A(S)$ is uniformly bounded.
To show equicontinuity, we compute the derivative of $(Ax)(t)$. Let $D = 1 - e^{-\int_0^\omega a(u)du}$. Then:

$$\begin{aligned}\frac{d}{dt}(Ax)(t) &= \frac{d}{dt}\left(-\int_{t-\omega}^{t} G(t,s)I(s)ds\right) \\ &= -\left[G(t,t)I(t) - G(t,t-\omega)I(t-\omega) + \int_{t-\omega}^{t} \frac{\partial}{\partial t}G(t,s)I(s)ds\right] \\ &= -\left[\frac{1}{D}I(t) - \frac{e^{-\int_0^\omega a(u)du}}{D}I(t-\omega) + \int_{t-\omega}^{t} (-a(t))G(t,s)I(s)ds\right] \\ &= -\frac{1}{D}\left[I(t) - e^{-\int_0^\omega a(u)du}I(t-\omega)\right] + a(t)(Ax)(t)\end{aligned}$$

Since $|I(t)| \leq \max_{t,s}|K(t,s)| \cdot \frac{1}{e} \cdot \tau_0$, $|a(t)|$ is bounded by continuity and periodicity, and $|(Ax)(t)|$ is bounded, there exists $L > 0$ such that:

$$\left|\frac{d}{dt}(Ax)(t)\right| \leq L$$

for all $t \in [0,\omega]$ and $x \in S$. Thus, $A(S)$ is equicontinuous.
By Arzela-Ascoli theorem, $A(S)$ is relatively compact.
Lemma 5. The operator $B: C_\omega \to C_\omega$ defined by $Bx = 0$ is a contraction mapping.
Proof. Trivially, $\|Bx - By\| = 0 \leq \lambda\|x - y\|$ for any $0 \leq \lambda < 1$.
Theorem 6 (Existence Theorem). Under assumptions (1)-(3) and if

$$M \cdot \omega \cdot \tau_0 \cdot \max_{t,s}|K(t,s)| < 1$$

then equation (1) has at least one ω-periodic solution.
Proof. We begin by defining the set $M = \{x \in C_\omega : \|x\| \leq R\}$, which constitutes a closed, convex, and nonempty subset of the Banach space C_ω. The constant R is selected to satisfy the inequality:

$$R \geq M \cdot \max_{t,s}|K(t,s)| \cdot \frac{1}{e} \cdot \omega \cdot \tau_0$$

To verify the first condition of Krasnoselskii's Fixed Point Theorem, we demonstrate that $Ax + By \in M$ for all $x, y \in M$. By the definition of our operators, we have $Ax + By = Ax + 0 = Ax$. To establish that $Ax \in M$, we must show that $\|Ax\| \leq R$. Employing the uniform bound derived in Lemma 4, we observe that for any $x \in M$:

$$\|Ax\| = \sup_{t\in[0,\omega]} |(Ax)(t)| \leq M \cdot \max_{t,s}|K(t,s)| \cdot \frac{1}{e} \cdot \omega \cdot \tau_0$$

Our initial selection of R ensures that $\|Ax\| \leq R$, thereby confirming that Ax remains within the set M. Furthermore, Lemma 4 establishes that the operator A is both compact and continuous. Additionally, Lemma 5 demonstrates that the operator B constitutes a contraction mapping.
Consequently, since all conditions of Theorem 1 are satisfied, there exists a fixed point $z \in M$ such that $z = Az + Bz = Az$. This fixed point z represents an ω-periodic solution to equation (1), which completes the proof.
Theorem 7 (Uniqueness Theorem). Under the same conditions as Theorem 6. the periodic solution is unique.
Proof. Suppose x and y are two fixed points. Then:

$$\begin{aligned}\|x - y\| &= \|Tx - Ty\| \\ &= \|Ax - Ay\| \\ &\leq M \cdot \max_{t,s}|K(t,s)| \cdot \|x - y\| \cdot \omega \cdot \tau_0 \\ &= \lambda\|x - y\|\end{aligned}$$

where $\lambda = M \cdot \omega \cdot \tau_0 \cdot \max_{t,s}|K(t,s)| < 1$. Thus, $\|x - y\| = 0$, so $x = y$. □

Conclusion

We have established the existence and uniqueness of periodic solutions for a class of nonlinear integro-differential equations with variable delay. The proof relies on the application of the Krasnoselskii Fixed Point Theorem, following the established methodology in the literature. By rigorously analyzing the boundedness and Lipschitz properties of the exponential nonlinearity, we derived a set of sufficient conditions on the equation's coefficients that guarantee a unique periodic solution. The presented approach provides a clear and robust framework for addressing this problem.

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